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Third Order Difference Methods for Hyperbolic Equations

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Third order difference equations for hyperbolic initial value problems have been developed in one and two space variables. Splitting methods in time and in space are used to achieve simplicity and economy in the algorithm. Example calculations are shown indicating the accuracy attainable.

1. INTRODUCTION

In recent years there has appeared in the literature a surge in the number of papers dealing with numerical solutions of partial differential equations. And, usually, the difference methods employed are of first or second order of accuracy. This restriction is not an arbitrary one, but rather, is related to the fact that computing machines have been relatively slow and their high speed memory capacity has been small; hence, a usable computational scheme must necessarily have the attribute of simplicity. In problems of more than one space dimension, even greater emphasis is placed on simplicity.

It is anticipated, however, that a new era of computability is almost upon us. We are referring to the use of parallel processors, i.e., N-serial type computing processors, each of which is synchronized and each of which can communicate with the other processors through a common memory or central controller. The value of N may be from 2^6 to 2^8 and the arithmetic speed of each individual processing unit will be in the sub-microsecond range. Through organizing the data, each mesh point or string of mesh points may have its own central processor, which means the solution on the entire mesh may be advanced essentially simultaneously. For such

a class of computing machines, the requirement of simplicity for the difference scheme may be relaxed.

In this note we propose a class of difference schemes for first order hyperbolic problems in one and two space dimensions. The methods are applicable to nonlinear initial value problems; these schemes are uniformly third order accurate in both the space variables and time and are similar to methods proposed by Strang [8].

2. A Third Order Difference Operator

We will construct a difference operator which is uniformly accurate to third order in each of the space and time increments Δx and Δt. There are schemes which are third or fourth order accurate in the space variable, but second order in the time step [10]. Such methods are, therefore, not uniformly third accurate in both the independent variables. The approximation scheme described here is constructed in divergent form—just as the original differential equation is written in divergent or conservation form. In order to describe the derivation, consider the differential equation in one dimension

$$\begin{aligned} u_t &= F(u, x, t, u_x) \\ u(x, 0) &= u_0(x) \qquad -\infty < x < \infty \end{aligned} \tag{2.1}$$

in which the flux is computed by evaluating F. Of course, for partial differential equations, F cannot be computed exactly since it depends on derivatives in the space variables. An approximate evaluation of F can be obtained if the space derivative, u_x, is replaced by a difference approximation δu. In this paper we look at Runge–Kutta type approximations to (2.1) which are third order accurate and for which the algorithm

$$u(t + \Delta t) = u(t) + \tfrac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$$\begin{aligned} k_1 &= \Delta t\, F(t_i, u_i) \\ k_2 &= \Delta t\, F(t_i + \Delta t/2, \quad u_i + k_1/2) \\ k_3 &= \Delta t\, F(t_i + \Delta t, \quad u_i - k_1 + 2k_2) \end{aligned}$$

is a third order Runge–Kutta method of integration for first order ordinary differential equations and is the basis for the method derived for partial differential equations. Since three evaluations of $F(t, u)$ are required, we would expect the same number of evaluations of difference approximations to the flux for the partial differential equation to be required. The form of the function F for the partial differential equation

$$u_t = f_x \equiv F \tag{2.2}$$

leads to the requirement that the approximation to F, $\tilde{F}$ differ only by terms of $O(\Delta t^3)$ so that if $\tilde{u}$ is a difference approximation to u satisfying

$$\tilde{u}(t + \Delta t) = \tilde{u}(t) + \Delta t \tilde{F}$$

then $| u(t + \Delta t) - \tilde{u}(t + \Delta t)| = O(\Delta t^4)$. Here

$$\tilde{F}(x_i , t_n , u_i , \delta u,...) = F(x, t, u, u_x ,...) + O(\Delta t^3).$$

The great advantage of Runge-Kutta methods (one example is Richtmyer's Lax–Wendroff two step method [1]) is that to achieve higher order accuracy in approximating $u(t)$ only repeated evaluations of F are required. For complicated F, however, this advantage becomes a disadvantage so that it may be desirable to use methods which do not require multiple evaluations of $F(x)$, having only slight changes in the value of the argument, x, of F.

As in ordinary differential equation theory it is possible to construct approximation techniques to partial differential equations using Taylor series methods. For high accuracy this requires evaluation of higher derivatives of (2.1) and for *coupled* systems of equations, i.e., for systems of the form (2.2), these derivatives become more and more complicated to evaluate. Lax and Wendroff [2] showed, however, that for systems of conservation laws given by (2.2), Taylor's method can be used to construct an elegant second order accurate algorithm. They showed that u_{tt} can be evaluated by using the original differential Eq. (2.2). If the matrix $A = \partial f/\partial u$ is introduced, then (2.2) can be written by using the chain rule as

$$u_t = A(u)\, u_x , \tag{2.2a}$$

so that

$$u_{tt} = f_{tx} = \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right)_x = (Af_x)_x \tag{2.3}$$

is found in terms of space derivatives only. If one wished to construct a third order method, it would be necessary to compute the time derivative of (2.3) for the next term in the Taylor series. Unfortunately, it is not possible to then eliminate terms containing $\partial A/\partial t = \dot{A}$ easily so that only space derivatives remain. It is clear that the dependence

$$\dot{A} = \dot{A}(u, u_x)$$

will occur through time differentiation of each term of A and back substitution for the time derivatives of u from the right hand side of (2.2) (where the function F is relatively simple). For the equations of gas dynamics, *this* procedure will result in unnecessarily complicated difference methods with the associated disadvantage that the form of the algorithm will not be conservative.[1]

[1] G. Zwas and S. Abarbanel have shown, however, for the scalar case where $A = a(u)$, that the l-th time derivative of u can be given by the compact expression $u_{tt} = (a^n u_x)_{nx}$, $n = l - 1$, which preserves the conservation form of the associated differential equation.

Instead of pursuing this approach, we use the alternate procedure which we first discussed, and which was first pointed out by Robert Richtmyer [1]. He showed that the Lax–Wendroff method could be written in two steps. For each step, only an evaluation of f_x is required—just what one would expect from a Runge–Kutta type method. The third order method which we now describe was first proposed by Rusanov, and we repeat some of the results that are contained in his paper [3]. We consider a sequence of iterates to the solution $u(t)$. The r-th iterate, $u^{(r)}$, defines an approximation to $\tilde{u}$, which is given by the system

$$u^{(r)} = u^{(0)} + \Delta t \sum_{s=0}^{r-1} \alpha_{rs} F(t_s, u^{(s)}), \qquad r = 1, 2, \ldots, R.$$

The function F is evaluated at time $t_s = t_0 + \tau_s \Delta t$, $s = 0, 1, \ldots R-1$, with $u(0) = \tilde{u}(t_0)$ defined at t_0 and $\tau_0 = 0$.

A solution $u^{(R)} = u(t_1)$ is obtained at $t_1 = t_0 + \Delta t$. To advance the solution from t_1 to t_2, the above procedure is repeated with t_0 replaced by t_1. The α_{rs} are determined by requiring that $u^{(R)}$ satisfy the Taylor expansion

$$u^{(R)} = u^{(0)} + \Delta t \left(\frac{\widetilde{\partial u^{(0)}}}{\partial t}\right) + \frac{\Delta t^2}{2!}\left(\frac{\widetilde{\partial^2 u^{(0)}}}{\partial t^2}\right) + \frac{\Delta t^3}{3!}\left(\frac{\widetilde{\partial^3 u^{(0)}}}{\partial t^3}\right) + O(\Delta t^4)$$

up to the required order of accuracy, which is three. The quantities in parentheses are difference approximations to the derivatives of u_0. To apply this procedure to the partial differential Eq. (2.2) it is convenient to write out the sequence of iterates using the notation $u_i^{(n)} = u(x_i, t_n)$ with $t_{n+1} = t_n + \Delta t$ and $\lambda = \Delta t/\Delta x$. We use the following spatial difference operators in the derivation

$$\begin{aligned} \mu f(x_i) &= \tfrac{1}{2}(f(x_{i+1/2}) + f(x_{i-1/2})) \\ \delta f(x_i) &= f(x_{i+1/2}) - f(x_{i-1/2}) \end{aligned} \tag{2.3}$$

and thus

$$\mu\delta f(x_i) = \tfrac{1}{2}(f(x_{i+1}) - f(x_{i-1})).$$

Then $u_i^{(1)} = \tilde{u}(x_i, t_n + \tau_1 \Delta t)$ is given by

$$u_{i+1/2}^{(1)} = \mu u_{i+1/2}^{(0)} + \alpha_{10}\{\lambda\delta f_{i+1/2}^{(0)}\}, \tag{2.4}$$

$u_i^{(2)} = \tilde{u}(x_i, t_n + \tau_2 \Delta t)$ is given by

$$u_i^{(2)} = u_i^{(0)} + \alpha_{20}\{\lambda\mu\delta f_i^{(0)}\} + \alpha_{21}\{\lambda\delta f_i^{(1)}\} \tag{2.5}$$

and $u_i^{n+1} = u_i^{(3)} = \tilde{u}(x_i, t_n + \Delta t)$ is given by

$$\begin{aligned} u_i^{n+1} = u_i^{(0)} &+ \alpha_{30}\{\lambda(I + \theta_{31}\delta^2)\,\mu\delta f_i^{(0)}\} \\ &+ \alpha_{31}\{\lambda(I + \theta_{32}\delta^2)\,\delta f_i^{(1)}\} + \alpha_{32}\{\lambda\mu\delta f_i^{(2)}\}. \end{aligned} \tag{2.6}$$

The sequence (2.4)–(2.6) is chosen to be in flux divergent form since the original differential equation is in this form. Equations (2.4) and (2.5) are generalizations of Richtmyer's two step method; it will be shown that this form will lead to a one parameter set of difference methods. Equation (2.6) represents a linear combination of central differences of the flux at the three previous time levels t_0, $t_0 + \tau_1 \Delta\tau$ and $t_0 + \tau_2 \Delta t$. The quantity $\mu u_{i+1/2}^{(0)}$ replaces $u_{i+1/2}^0$ for stability of (2.4) (and indeed for stability of $u_i^{(2)}$).

The prescription to find the α_{rs} is to use operators (2.3) in (2.4)–(2.6), and then expand each term in the brackets as a Taylor series; for example,

$$\begin{aligned} \mu f_i &\equiv f_i + \tfrac{1}{2} d_x^2 f_i (\Delta x^2/2) \\ \delta f_i &\equiv d_x f_i\, \Delta x + \tfrac{1}{6} d_x^3 f_i (\Delta x^3/8) \\ \mu\delta f_i &\equiv d_x f_i\, \Delta x + \tfrac{1}{6} d_x^3 f_i\, \Delta x^3. \end{aligned} \tag{2.7}$$

We use the symbol [$\equiv$] to indicate that the expressions (2.7) are correct modulo terms $O(\Delta x^4)$. In order to compare the resulting expression with the third order expansion

$$\begin{aligned} u^{n+1} &= u^n + (d_t u^n)\,\Delta t + (d_{tt} u^n)\frac{\Delta t^2}{2} + (d_{ttt} u^n)\frac{\Delta t^3}{6} + O(\Delta t^4) \\ &= u^n + (d_x f^n)\,\Delta t + (d_{tx} f^n)\frac{\Delta t^2}{2} + (d_{ttx} f^n)\frac{\Delta t^3}{6} + O(\Delta t^4), \end{aligned} \tag{2.8}$$

the functions $f^{(1)}(t_0 + \tau_1 \Delta t)$ and $f^{(2)}(t_0 + \tau_2 \Delta t)$ must be expressed as Taylor expansions about $f^n = f^{(0)}(t_0)$. For instance, since we want to find $\mu\delta f_1^{(2)}$, first use (2.7) and then apply Taylor's formula to the result to obtain

$$\mu\delta f_i^{(2)} \equiv (d_x f_i^{(0)})\,\Delta x + \tau_2\,\Delta t (d_{xt} f_i^{(0)})\,\Delta x + \frac{(\tau_2\,\Delta t)^2}{2}(d_{xtt} f_i^{(0)})\,\Delta x + \frac{1}{6}(d_{xxx} f_i^{(0)})\,\Delta x^3.$$

In a similar fashion, we obtain the required expansion for each of the bracketed expressions in equation (2.6). The expression for u^{n+1} results in

$$\begin{aligned} u^{n+1} = u^n &+ (\alpha_{30} + \alpha_{32})(d_x f)\,\Delta t + \alpha_{32}\tau_2 (d_{xt} f)\,\Delta t^2 \\ &+ \frac{\alpha_{32}\tau_2^2}{2}(d_{xtt} f)\,\Delta t^3 \\ &+ \frac{1}{6}(\alpha_{30} + 6\alpha_{30}\theta_{31} + \alpha_{32})\, d_{xxx} f\,\Delta t\,\Delta x^2. \end{aligned} \tag{2.9}$$

The terms containing α_{31} are at most second order accurate so to preserve the accuracy of u^{n+1}, $\alpha_{31} = 0$.

Comparing (2.8) with (2.9) we see that for u^{n+1} to be third order accurate, Eq. (2.6) must have coefficients α_{3s} and θ_{31} which satisfy

$$\begin{aligned} \alpha_{30} + \alpha_{32} &= 1, \qquad \alpha_{32}\tau_2 = \frac{1}{2}; \\ \alpha_{30} + 6\alpha_{30}\theta_{31} + \alpha_{32} &= 0, \qquad \frac{\alpha_{32}\tau_2^2}{2} = \frac{1}{6}. \end{aligned} \tag{2.10}$$

These relations imply that $\alpha_{32} = 3/4$, $\alpha_{30} = 1/4$, $\tau_2 = 2/3$ and $\theta_{31} = -2/3$. Again, by using this expansion procedure on the bracketed expressions in Eq. (2.5), we obtain

$$u^{(3)} = u^n + \alpha_{20}(d_x f)\,\Delta t + \alpha_{21}((d_x f)\,\Delta t + (d_{xt} f)\,\tau_1\,\Delta t^2). \tag{2.11}$$

For the above equation for $u^{(3)}$ to differ from the Taylor expansion for $u^{(3)}$ about u^n by only terms of $O(\Delta t^3)$, the α_{2s} must satisfy

$$\alpha_{20} + \alpha_{21} = \tau_2, \qquad \alpha_{21}\tau_1 = \tfrac{1}{2}\tau_2^2. \tag{2.12}$$

Similarly, for Eq. (2.4) to yield first order accurate data for $u^{(1)}$,

$$\alpha_{10} = \tau_1. \tag{2.13}$$

Hence, we have specified the coefficients to within one parameter, namely, τ_1.

For $\tau_1 = 1/3$, we have $\alpha_{20} = 0$ and $\alpha_{21} = 2/3$. In this case the difference equations become

$$u_{i+1/2}^{(1)} = \frac{1}{2}(u_{i+1}^n + u_i^n) + \frac{1}{3}\{\lambda(f_{i+1}^n - f_i^n)\] \tag{2.14a}$$

$$u_i^{(2)} = u_i^n + \frac{2}{3}\{\lambda(f_{i+1/2}^{(1)} - f_{i-1/2}^{(1)})\} \tag{2.14b}$$

$$\begin{aligned} u_i^{n+1} = u_i^n + \frac{1}{4}\Big\{&\frac{\lambda}{6}(-2f_{i+2}^n + 7f_{i+1}^n - 7f_{i-1}^n + 2f_{i-2}^n) \\ &+ \frac{3}{4}\Big\{\frac{\lambda}{2}(f_{i+1}^{(2)} - f_{i-1}^{(2)})\Big\}. \end{aligned} \tag{2.14c}$$

These equations are the analog of the integral of the conservation law. To see this we use Fig. 1 to define integral quantities U_0 and U_1:

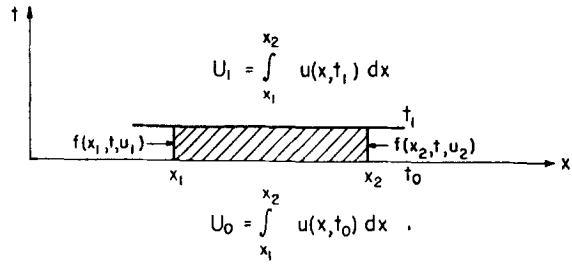

FIG. 1. The Region of space-time domain over which the conservation laws, Eq. (2.2), are integrated.

Integrate (2.2) over the shaded region in the space time domain to obtain

$$U_1 = U_0 + \int_{t_0}^{t_1} f(x_2, t, u_2)\,dt - \int_{t_0}^{t_1} f(x_1, t, u_1)\,dt.$$

In the above expression $u_1(t)$ and $u_2(t)$ are the values of u at the net points x_1 and x_2; the time dependency is indicated. The values of u computed by system (2.14) yield a sequence $u(t_i)$, $t_0 \leqslant t_i \leqslant t_1$, which allows the integrals of the flux over the time interval $t_1 - t_0$ to be approximated more accurately. The sums U_1, U_0 are seen to be telescoping sums in f cancelling in pairs over all net points between x_1 and x_2.

We show in the next section the stability properties of system (2.14); indeed, (2.14) is unconditionally unstable (after all this effort!). If the right hand side of (2.14c) is denoted by R^n, then a stable scheme is obtained by subtracting an undivided difference quotient of fourth order from R^n, i.e.,

$$u_i^{n+1} = R^n - \frac{\omega}{24}\,\delta^4 u_i^n, \qquad \omega \geqslant 0. \tag{2.15}$$

The net point cluster of the difference scheme (2.14) is shown in Fig. 2.

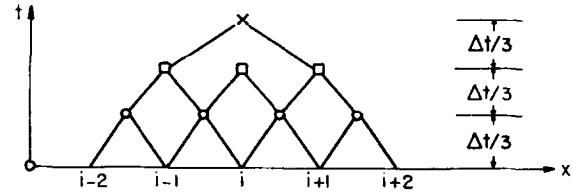

FIG. 2. Net point cluster for difference Eqs. (2.14a, b, c) and (2.15).

The points designated ○ are first order accurate data, □-points are second order accurate data, and the ×-point is the third order accurate solution. The schematic indicates the operator defined by (2.14a) is applied five times, (2.14b) is applied three times, and (2.14c) and (2.15) are applied once. Another scheme is obtained for $\tau_1 = 2/3$ ($\alpha_{20} = \alpha_{21} = 1/3$); for this choice of τ_1 the difference scheme is given by

$$u^{(1)}_{i+1/2} = \frac{1}{2}(u^n_{i+1} + u^n_i) + \frac{2}{3}\{\lambda(f^n_{i+1} - f^n_i)\} \tag{2.16a}$$

$$u^{(2)}_i = u^n_i + \frac{2}{3}\left\{\frac{\lambda}{2}\left[(f^{(1)}_{i+1/2} - f^{(1)}_{i-1/2}) + \frac{1}{2}(f^n_{i+1} - f^n_{i-1})\right]\right\} \tag{2.16b}$$

and Eqs. (2.14c) and (2.15).

The net point cluster of this difference scheme is also shown in Fig. 3.

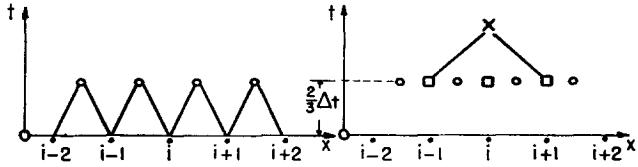

FIG. 3. Net point cluster for difference Eqs. (2.16a, b), (2.14c), and (2.15).

Since (2.14c) requires second order data $u^{(2)}_i$, as well as initial data, $u^{(0)}_i$ for evaluation, in principle any second order difference operator can be used to generate this data; in particular $u^{(2)}_i$ could be obtained from

$$u^{(2)}_i = u^n_i - \frac{\lambda}{3}(f^n_{i+1} - f^n_{i-1}) + \frac{2\lambda^2}{9}\{A_{i+1/2}(f_{i+1} - f_i) - A_{i-1/2}(f_i - f_{i-1})\}$$

which is the Lax–Wendroff method.

In addition, a simple method can be used to obtain first order accurate data $u^{(1)}_i$ other than (2.14a) or (2.16a). The only requirement is that the overall scheme be stable. In choosing $\alpha_{10} = \tau_1$ we have considered only values which result in positive weights in (2.5) and a value for the time step of (2.4) which is less than or equal to the time step of (2.5). The permissible range of α_{10} required to satisfy these conditions is $1/3 \leqslant \alpha_{10} \leqslant 2/3$.

3. Stability of the One Dimensional Difference Operator

Let M be the amplification matrix obtained by first letting $f(u) = Au$ and then substituting (2.14a) and (2.14b) into (2.14c) subject to the viscosity expression

(2.15). The amplification matrix of this combined system, obtained by substituting $u_0 \exp(i\kappa x)$ for u_i^n is

$$M(\xi, \lambda, \omega) = I - \frac{\lambda^2 A^2}{2} \sin^2 \xi - \frac{\omega}{6}(1 - \cos \xi)^2 + i\lambda A \sin \xi \left(1 + \frac{1}{3}(1 - \cos \xi)(1 - \lambda^2 A^2)\right) \tag{3.1}$$

where $\xi = \kappa \Delta x$ and $0 \leqslant \xi \leqslant \pi$.

Call m the eigenvalues of M (see Fig. 4) and $\sigma = \lambda\rho$, ρ the eigenvalues of A. Construct the function $g(\xi, \sigma, \omega)$ from the real, R, and imaginary, I, parts of (3.1) by

$$\begin{aligned} g &= R^2 + I^2 - 1 \\ &= |m^2| - 1. \end{aligned} \tag{3.2}$$

Absolute value of the eigenvalues of Equation (3.1) showing dependence on ω. The values of σ are within .05 of the maximum allowable for each ω.

Curve number	ω	σ
0	0	1.00
1	1/3	0.05
2	2/3	0.15
3	3/3	0.25
4	5/3	0.45
5	8/3	0.80
6	9/3	0.95
7	10/3	1.00

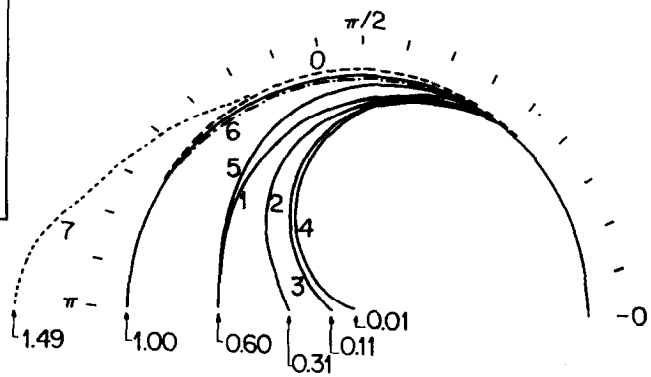

FIG. 4. Spectral radius of the right hand side of Eq. (3.1) showing dependence on ω. The values of σ are within 0.05 of the maximum allowable for each ω.

Since it takes so much effort to compute g, we shall state its value here.

$$\begin{aligned} g(\xi, \sigma, \omega) = {} & \frac{\sigma^4}{4} \sin^4 \xi + \frac{\sigma^2}{3}\left(\frac{\omega}{2} + \frac{(1 - \sigma^2)^2}{3}\right) \sin^2 \xi (1 - \cos \xi)^2 \\ & + \frac{\omega^2}{36}(1 - \cos \xi)^4 - \frac{\omega}{3}(1 - \cos \xi)^2 \\ & + \frac{2}{3}\sigma^2(1 - \sigma^2) \sin^2 \xi (1 - \cos \xi). \end{aligned} \tag{3.2a}$$

Now $|m^2| < 1$ if, and only if, $g < 0$. Allow $\xi = \pi$ and observe that

$$g(\pi, \sigma, \omega) = \tfrac{4}{9}\omega(\omega - 3) \leqslant 0 \tag{3.3}$$

if, and only if, $0 \leqslant \omega \leqslant 3$.

For small values of ξ, g can be written as

$$m(\xi, \sigma, \omega) = e^{i\sigma\xi} + (4\sigma^2 - \sigma^4 - \omega)\frac{\xi^4}{4!} + O(\xi^5). \tag{3.4}$$

Geometrically, one sees that g will exceed one, in the complex plane, unless

$$\omega \geqslant 4\sigma^2 - \sigma^4. \tag{3.5}$$

That is, the operator (2.14) will not be stable. Hence combining (3.3) and (3.5) we conclude that $4\sigma^2 - \sigma^4 \leqslant \omega \leqslant 3$ which in turn implies that $0 \leqslant \sigma^2 \leqslant 1$, $3 \leqslant \sigma^2 \leqslant 4$. To show that σ^2 takes on allowable values only in the interval $0 \leqslant \sigma^2 \leqslant 1$, which is more strict than the Courant–Friedrichs–Lewy condition [4], it is necessary to show that for any value of σ such that $3 \leqslant \sigma^2 \leqslant 4$, there exists a $\xi = \xi_0 \neq 0$ such that $g(\xi_0, \sigma, \omega) \geqslant 0$ for ω given by (3.5). In equation (3.2a) set $\xi = \pi/2$:

$$\begin{aligned} g\left(\frac{\pi}{2}, \sigma, \omega\right) &= \frac{1}{36}(\omega^2 + \omega(6\sigma^2 - 12) + (4\sigma^6 - 23\sigma^4 + 28\sigma^2)) \\ &\geqslant \frac{1}{36}((4\sigma^2 - \sigma^4)^2 + (4\sigma^2 - \sigma^4)(6\sigma^2 - 12) + (4\sigma^6 - 23\sigma^4 + 28\sigma^2)) \\ &= \frac{\sigma^2}{36}(\sigma^2 - 1)(\sigma^2 - 4)(\sigma^2 - 5) \geqslant 0 \qquad \text{for} \quad 3 \leqslant \sigma^2 \leqslant 4. \end{aligned}$$

We see $g < 0$ only if $0 \leqslant \sigma^2 \leqslant 1$.

It is clear that if $\omega = 4\sigma^2 - \sigma^4$ and if $0 < \sigma < 1$, then $g < 0$. Setting

$$g_*(\xi, \sigma) = g(\xi, \sigma, 4\sigma^2 - \sigma^4),$$

and noting that $\sin^2 \xi = (1 - \cos \xi) \cdot (1 + \cos \xi)$, we compute

$$g_*(\xi, \sigma) = \frac{\sigma^2}{36}(1 - \cos \xi)^2 (4 - \sigma^2)(1 - \sigma^2) \cdot P_2(\cos \xi), \tag{3.6}$$

where the quadratic polynomial P_2 is defined by

$$\begin{aligned} P_2(x) &= ax^2 + bx + c \\ a &= -(1 - \sigma^2) \\ b &= 2(3 - \sigma^2) \\ c &= -(5 - \sigma^2). \end{aligned} \tag{3.7}$$

It can be shown that $P_3(x) < 0$ if $x < 1$, which implies $P_3(\cos \xi) < 0$ if $0 < \xi \leqslant \pi$. Clearly, $g_*(\xi, \sigma) < 0$ since $0 < \sigma^2 < 1$. One also observes that $g_*(0, \sigma) = 0$. At this point we have shown that the right side of (3.2) is negative definite for $\pi \geqslant \xi > 0$ and

$$\begin{aligned} &\text{(i)} \quad 0 < \sigma < 1 \\ &\text{(ii)} \quad \omega = 4\sigma^2 - \sigma^4. \end{aligned} \tag{3.8}$$

We exclude the case where $\sigma = 1$, since if that occurs, $\omega = 3$, and $g(\xi, 1, 3) = 1$, which leads to $|m^2| = 1$, i.e., the associated difference operator is not dissipative.

We look at the quantity $|m|$. It can be bounded (using (3.4)) for small ξ:

$$|m| \leqslant 1 + \left(\frac{4\sigma^2 - \omega - \sigma^4}{2}\right) \frac{\xi^4}{4!} + O(\xi^5). \tag{3.9}$$

One can show that for any $\sigma \in (0,1)$ there exists an $e > 0$ such that if $\omega = 4\sigma^2 - \sigma^4 + e$ and $0 < \xi \leqslant \pi$, $g(\xi, \sigma, \omega) < 0$, which is equivalent to $|m| < 1$. There exists a pair (δ_1, η), each greater than zero, such that $0 \leqslant \xi \leqslant \eta$, implies $|m| \leqslant 1 - \delta_1 \xi^4$. Since $[\eta, \pi]$ is compact and since $|m| < 1$ on $[\eta, \pi]$, there exists a positive δ_2 such that if $\eta \leqslant \xi \leqslant \pi$, then $|m| \leqslant 1 - \delta_2 \xi^4$. Let $\delta =$ minimum (δ_1, δ_2); we see that the difference scheme associated with the amplification matrix (3.1) is dissipative for $0 \leqslant \xi \leqslant \pi$ in the sense of Kreiss since

$$|m| \leqslant 1 - \delta\xi^4, \qquad \delta > 0. \tag{3.10}$$

Since the accuracy is of order 3, (3.1) is stable [5].

4. Two Dimensional Methods

For two dimensional hydrodynamic flows in which x and y are the cartesian coordinates, the equations of motion can be written in conservation-law form

$$u_t = f_x + g_y, \tag{4.1}$$

where g is the vector representing the flux of the mass, momentum and energy per unit volume in the y direction. We carry out differentiation of (4.1) using the chain rule to obtain

$$u_t = A(u)\, u_x + B(u)\, u_y. \tag{4.2}$$

In general the matrices A and B do not commute and are not normal.

If one considers the class of linear problems where $A_0 = A(u_0)$ and $B_0 = B(u_0)$, u_0 the state about which the motion is linearized, then (4.2) may be integrated to yield

$$u(t + \Delta t) = Pu(t),$$
$$P = \exp\left[\left(A_0 \frac{\partial}{\partial x} + B_0 \frac{\partial}{\partial y}\right) \Delta t\right]. \tag{4.3}$$

Equation (4.3) is also valid even if A_0 and B_0 vary; however, the variation must be independent of time. With the obvious change in notation P may be written as

$$P = e^{A+B}. \tag{4.4}$$

This operator is called the exact solution operator of equation (4.2). The Fourier transform of P, $\hat{P} = \exp(i(A\xi + B\eta))$ is called the symbol of the operator P (see Ref. [5]). By multiplying the initial data $u(x, y, 0) = u(0)$, r successive times using the operator P, we can map $u(0)$ into $u(T)$, $T = r\,\Delta t$.

In forming difference approximations to (4.1) or equivalently in approximating the operator P, the question of stability arises. The analysis of stability of the difference operator becomes difficult, especially as the order of accuracy (and corresponding complexity) of the difference scheme increases. Indeed, in Strang's paper [6] on the construction of accurate difference methods, he is motivated in the construction of difference methods by approximating P to the desired degree of accuracy by operators which are products of e^A and e^B.

It is well-known, for instance, that a first order approximation to the matrix P can be written as

$$P = e^A e^B + O(\Delta t^2) \tag{4.5}$$

since A and B are of order Δt. The error in (4.5) goes to zero when A and B are scalars. The operators e^A and e^B can be thought of as exact solution operators to the one dimensional differential equation of the form (2.2a) defined separately for the x and y directions. Let $L(A)$ and $L(B)$ be difference approximations to the operators e^A and e^B, respectively. If $\hat{L}(A)$ is the symbol of the x-difference operator and

$$|\hat{L}(A) - e^{iA\xi}| = O(\Delta t^{p+1}), \tag{4.6}$$

then we conclude that the difference operator is accurate to order p.

Strang has shown [6], [7], and [8] that if one considers an operator $L_1(A, B)$ formed from the product of the one dimensional operators

$$L_1(A, B) = L(A)\,L(B)$$

then

$$| \tfrac{1}{2}(\hat{L}_1(A, B) + \hat{L}_1(B, A)) - \hat{P} | = O(\xi^3, \eta^3), \tag{4.7}$$

where $\hat{P}$ has been defined previously. Strang has also noted recently [8] that it is possible to satisfy (4.7) with the product

$$L(A/2)\, L(B)\, L(A/2), \tag{4.8}$$

replacing the sum in (4.7); hence (4.8) provides the structure for another difference scheme of second order accuracy.

The stability of (4.8) follows immediately from the stability of each one dimensional operator $L(A)$ and $L(B)$. For (4.8) to be second order accurate and stable each one dimensional operator, given by

$$L(A/2) = I - (\lambda/2)\, A\delta + (\lambda^2/8)\, A^2\delta^2, \tag{4.9}$$

which is the Lax-Wendroff operator, would be stable if

$$\begin{aligned} \lambda\mu(A) &\leqslant 1 \\ \lambda\mu(B) &\leqslant 1. \end{aligned} \tag{4.10}$$

Equation (4.8) can be used with a second order two step procedure rather than (4.9), i.e., Eqs. (2.14a) and (2.14b) or system (2.16) (with the appropriate time step). The advantage is the elimination of the evaluation of the matrix A in the difference scheme.

Gourlay and Morris [9] have performed some computations with such schemes. They have adopted the operator in (4.7) for practical computations by using two step versions of $L(A)$ and $L(B)$.

We wish to look for difference schemes of the form given by (4.8) which are of uniform third order accuracy. The structure of the difference operator will then be based on the third order one dimensional operators discussed in Section 2.

We have considered generalizations, of the operators given in (4.7) and (4.8), of the form

$$S = \sum_j c_j \prod_i e^{\alpha_{ij}A} e^{\beta_{ij}B}, \tag{4.11}$$

where $\sum_j c_j = 1$ and $\sum \alpha_{ij} = \sum \beta_{ij} = 1$ with each $\alpha, \beta \geqslant 0$.

Clearly, if one chooses $c_1 = 1$ with $\alpha_{11} = \beta_{11} = \beta_{21} = \alpha_{31} = 1/2$ and $\alpha_{21} = 0 = \beta_{31}$, then (4.11) becomes

$$S_1(A, B) = e^{A/2} e^{B} e^{A/2}. \tag{4.12}$$

With the constants $c_1 = 1/2$ and $\alpha_{11} = \beta_{11} = 1$, and $c_2 = 1/2$ and $\alpha_{12} = 0 = \beta_{22}$, $\beta_{12} = 1 = \alpha_{22}$, (4.11) becomes

$$S_2 = \tfrac{1}{2}(e^A e^B + e^B e^A). \tag{4.13}$$

The operators S_1 and S_2 are the operators Strang has investigated. It is interesting to consider the operator formed from linear combinations of (4.12) and (4.13), i.e.,

$$S_3 = \frac{4}{3}\left(\frac{S_1(A, B) + S_1(B, A)}{2}\right) - \frac{1}{3} S_2 . \tag{4.14}$$

If the one dimensional differential operators in (4.14) are replaced by corresponding one-dimensional difference operators defined by

$$L(A) = I + \lambda A\mu\delta + \frac{\lambda^2}{2} A^2\delta^2 + \frac{\lambda^3}{6} A^3\delta^2\mu\delta, \tag{4.15}$$

then it may be verified by direct computation that the resulting difference approximation L_3 to the differential operator S_3 satisfies

$$L_3 = I + \lambda(A + B) + (\lambda^2/2)(A + B)^2 + (\lambda^3/3!)(A + B)^3. \tag{4.16}$$

Here, for simplicity we have used the abbreviations $A = A(u)\,\delta_x$, $B = B(u)\,\delta_y$ for the centered difference operators and $\lambda = \Delta t/\Delta x$, $\Delta = \Delta x = \Delta y$. This is precisely the expansion for (4.4) up to cubic terms. The operator, S_3, was first found by J. Dunn.

The procedure used to derive the third order approximation to (4.4), which is in some sense computationally optimal, follows.

The operator defined by (4.14) is complicated and inefficient as it requires ten sweeps through the mesh—five in the x direction and five in the y direction—to advance the solution one time step. It is clear that more compact forms resulting in economical algorithms suggested by (4.11) are desirable. Consider the differential operator

$$c_1 e^A e^B + c_2 e^{\alpha A} e^{\beta B} e^{(1-\alpha)A} e^{(1-\beta)B}. \tag{4.17}$$

If the constants are chosen correctly (4.17) can be made to differ from (4.4) by terms of $O(\Delta t^4)$. To do this first expand each of the exponential forms up to terms involving cubic powers of the matrices A and B. This allows the evaluation of the term corresponding to the coefficient c_2:

$$\begin{aligned} &I + (A + B) + \tfrac{1}{2}(A^2 + B^2) + [\alpha + (1 - \alpha)(1 - \beta)]\, AB \\ &\quad + [\beta(1 - \alpha)]\, BA + \tfrac{1}{6}(A^3 + B^3) \\ &\quad + \tfrac{1}{2}[\alpha^2 + 2\alpha(1 - \alpha)(1 - \beta) + (1 - \alpha)^2(1 - \beta)]\, A^2B \\ &\quad + \alpha\beta(1 - \alpha)\, ABA + \tfrac{1}{2}\beta(1 - \alpha)^2\, BA^2 + \tfrac{1}{2}\beta^2(1 - \alpha)\, B^2A \\ &\quad + \beta(1 - \alpha)(1 - \beta)\, BAB + \tfrac{1}{2}[\alpha\beta^2 + 2\alpha\beta(1 - \beta) + (1 - \beta)^2]\, AB^2, \end{aligned} \tag{4.18}$$

and the evaluation of the term corresponding to the coefficient c_1:

$$I + (A + B) + \tfrac{1}{2}(A^2 + B^2) + AB + \tfrac{1}{6}(A^3 + B^3) + \tfrac{1}{2}(A^2B + AB^2). \quad (4.19)$$

For simplicity we just equate the coefficients of the matrices BA, ABA and BAB in (4.18) to their proper values:

$$c_2\beta(1 - \alpha) = \tfrac{1}{2},$$

$$c_2\alpha\beta(1 - \alpha) = \tfrac{1}{6},$$

and

$$c_2\beta(1 - \alpha)(1 - \beta) = \tfrac{1}{6}.$$

These equations yield the values $\alpha = 1/3$, $\beta = 2/3$, $c_2 = 9/8$ and $c_1 = -1/8$. We again observe the appearance of the nonpositive weight in our difference scheme. The difference equation becomes

$$L_4 = (9/8)\, L(A/3)\, L(2B/3)\, L(2A/3)\, L(B/3) - \tfrac{1}{8}L(A)\, L(B). \quad (4.20)$$

Each one dimensional difference operator in (4.20) is defined by (4.15). The proof of stability of (4.20) (and that of (4.14)) does *not* follow from the fact that the norm of each one dimensional operator $\| L \| < 1$. If each coefficient c_i in (4.17) were greater than zero, then L_4 would be a convex operator and one could conclude in that case that L_4 had norm less than one. We defer this question until later.

It appears that (4.20) is most efficient in the sense that the number of one dimensional sweeps is a minimum for a third order operator. One needs at least six applications of the exponential operators e^A and e^B to match the noncommutative terms that result from the third order term in the Taylor expansion for e^{A+B}; i.e., $\frac{1}{6}(A + B)^3$. The proof of this statement involves consideration of linear combinations of products of e^A and e^B taken two at a time and three at a time. All such combinations fail to yield simultaneously the matrix operators ABA and BAB. Next, consider product combinations of the one dimensional operators taken four at a time:

$$e^{\alpha A}e^{\beta B}e^{(1-\alpha)A}e^{(1-\beta)B}.$$

Expanding and considering the requirement that second order accuracy implies $\beta(1 - \alpha) = 1/2$, we find that for third order accuracy $\alpha = 1/3$, $\beta = 2/3$. Hence there is a contradiction. Finally operators formed from products taken five at a time are of the general form

$$e^{\alpha_1 A}e^{\beta_1 B}e^{\alpha_2 A}e^{\beta_2 B}e^{\alpha_3 A}. \quad (4.21)$$

For (4.21) to be third order accurate α_1 must satisfy $12\alpha_1^2 - 6\alpha_1 + 1 = 0$. This polynomial however has only complex roots.

Hence a third order splitting method of the form (4.11) must have at least six terms. We state that the linear combination of the form (4.12), i.e.,

$$c_1 e^{\alpha A} e^{B} e^{(1-\alpha)A} + c_2 e^{\beta B} e^{A} e^{(1-\beta)B}$$

cannot differ from e^{A+B} by terms of $O(\Delta t^4)$. Satisfying consistency requires that α and β must satisfy

$$\tfrac{1}{2}(\alpha - \beta)(\alpha + \beta - 1) = 0.$$

If $\alpha = \beta$ we can show that α must satisfy a quadratic in α with complex roots. If $\alpha + \beta = 1$, α satisfies $(\alpha - 1/2)(\alpha - 1) = 0$. Now $\alpha \neq 1$ so $\alpha = 1/2$ and therefore $\beta = 1/2$ which is a contradiction.

If one were to consider only operators with $c_i > 0$, then third order accuracy could still be obtained but with a relaxation of the condition that all α_i, $\beta_i > 0$. Although the analysis of stability would be trivial, one would have to accept multistep difference methods with operators having a negative time step. For flows which contain shocks or other irreversible phenomena the problem is not well posed. If the flow is smooth and thermodynamically reversible there may be no drawback to such methods. We indicate in section 7 some results using (4.20).

5. Asymptotic Operators

It is possible to generate a difference operator with α_{ij}, β_{ij} and $c_i \geqslant 0$, but only asymptotically. Consider the differential operator

$$S(A, B; N) \equiv e^{A/N} e^{2B/N} e^{A/N}$$

and its conjugate $S(B, A; N)$. Then

$$S_N = (S(A, B; N)\, S(B, A; N))^{N/4}, \qquad N = 4, 8, \ldots \tag{5.1}$$

is called an asymptotic third order difference operator. S_N would be an exact third order operator if the coefficient δ_1 of the terms (A^2B, B^2A), (BA^2, AB^2) and the coefficient δ_2 of the terms (ABA, BAB) satisfied $\delta_1 = \delta_2 = 1/6$. Instead these coefficients are functions of N. We have computed bounds on the coefficients and show them for several values of N:

N	$\lvert \delta_1 - 1/6 \rvert <$	$\lvert \delta_2 - 1/6 \rvert <$
4	0.0053	0.0105
8	0.0013	0.0026
12	0.0006	0.0012

It appears, that using (5.1),

$$| \lim_{N\to\infty} S_N - P | = O(\Delta t^4).$$

Since $| S(A, B)| < 1$ and $| S(B, A)| < 1$, then $| S_N | < 1$ which shows the stability of (5.1). The operator defined by (5.1) achieves its accuracy by using finer and finer time steps, $\Delta t/N$, as $N \to \infty$; indeed it is the form assumed for S_N that gives the rapid convergence of coefficients δ_1, δ_2. The operator $(e^{A/N}e^{B/N})^{N/2}$, $N = 2, 4,...$ will also give asymptotic high order accuracy (greater than first order) but requires many evaluations (large N) per time step. In comparison, (5.1) may be satisfactory for $N = 4$.

6. Stability of Two Dimensional Operators

Except for the brief discussion on the stability of asymptotic operators, we have not found a satisfactory method for the analysis of the stability of the operators given by (4.11). Our only recourse is to carry out a *numerical* analysis of the eigenvalues of the amplification matrix using the digital computer. We have completed a calculation in which the independent variables are the wave numbers (ξ, η) in (x, y) space. We took the dissipation coefficient ω to be $\omega = 4\sigma^2 - \sigma^4 + \epsilon$ with $-0.2 \leqslant \epsilon \leqslant 0.2$ in steps of 0.1 and with $0 \leqslant \sigma \leqslant 1$ also in steps of 0.1.

From this parametric study, it appears that the spectral radius of $(\hat{L}_4\xi, \eta)$, the transform of (4.20), satisfies

$$| \mu(\hat{L}_4(\xi, \eta))| < 1 \tag{6.1}$$

and

$$| \mu(\hat{L}_4(\xi, \eta))| \leqslant 1 - \delta \,|\, \theta \,|^4, \delta > 0 \tag{6.2}$$

where $\theta = (\xi^2 + \eta^2)^{1/2}$ is the L_2 norm in wave number space, if

$$\begin{aligned} &\text{(i)} && 0 < \sigma < 1, && \epsilon \geqslant 0 \\ &\text{(ii)} && \sigma = 0, && \epsilon > 0. \end{aligned} \tag{6.3}$$

However, if $0 < \sigma < 1$ and $\epsilon < 0$, the spectral radius exceeds one. Indeed, if $\epsilon = 0.1$ we can choose, in (6.2), $\delta = 10^{-4}$ uniformly independent of σ. To achieve this define

$$\delta' \equiv \frac{1 - \mu(\xi, \eta)}{\theta^4},$$

then pick

$$\delta = \inf_{(\xi,\eta)} \delta'.$$

We have found that δ' is smallest when (ξ, η) is near (π, π). For $\xi = \eta$ we indicate the behavior of $|\mu(\xi, \xi)|$ in Table I.

We have defined the distance from the origin $r = \sqrt{2}\,\xi$ in wave number space and $\epsilon = \omega - (4\sigma^2 - \sigma^4)$. In the next section we present further evidence as to the usefulness of these third order operators.

TABLE I

	$(\lvert \mu(\hat{S}_3(\xi, \xi))\rvert - 1)$		
r	$\epsilon = -0.1$	$\epsilon = 0$	$\epsilon = +0.1$
0.0	0.0	0.0	0.0
0.01	2.9×10^{-11}	-1.6×10^{-11}	-6.0×10^{-11}
0.10	2.9×10^{-7}	-1.6×10^{-7}	-6.0×10^{-7}
0.5	1.7×10^{-4}	-9.7×10^{-5}	-3.7×10^{-4}
1.0	2.6×10^{-3}	-1.5×10^{-3}	-6.5×10^{-3}
2.0	3.2×10^{-2}	-1.9×10^{-2}	-6.7×10^{-2}
3.0	1.0×10^{-1}	-6.2×10^{-2}	-2.1×10^{-1}

7. Results

We describe some numerical experiments carried out with the scheme (2.14) for the Riemann problem in one dimension and with (4.20) for a two dimensional scalar problem invented by Crowley [10].

Figures 5 and 6 show the results of two calculations using system (2.14) and (2.15) to obtain approximate solutions to (2.2). Both calculations start with the same initial data, i.e., two constant states separated by a discontinuity:

$$\left.\begin{aligned} \rho^{-1}(x) &= 2.0 \\ u(x) &= 0.0 \\ p(x) &= 0.571 \end{aligned}\right\} x \geqslant 0 \qquad \left.\begin{aligned} \rho^{-1}(x) &= 2.245 \\ u(x) &= 0.698 \\ p(x) &= 3.528 \end{aligned}\right\} x < 0$$

In Figure 5, $\sigma = 0.9$ with $\omega = 0.75$. The instability is clearly shown. Figure 6

scalar problem. The differential equation

$$r_t + ur_x + vr_y = 0 \tag{7.1}$$

describes the motion of the function $r(x, y, t)$ in the $x - y$ plane if the velocity components u and v are specified. We take them to be

$$\begin{pmatrix} u \\ v \end{pmatrix} = |\,\Omega\,| \begin{pmatrix} -y \\ x \end{pmatrix} \tag{7.2}$$

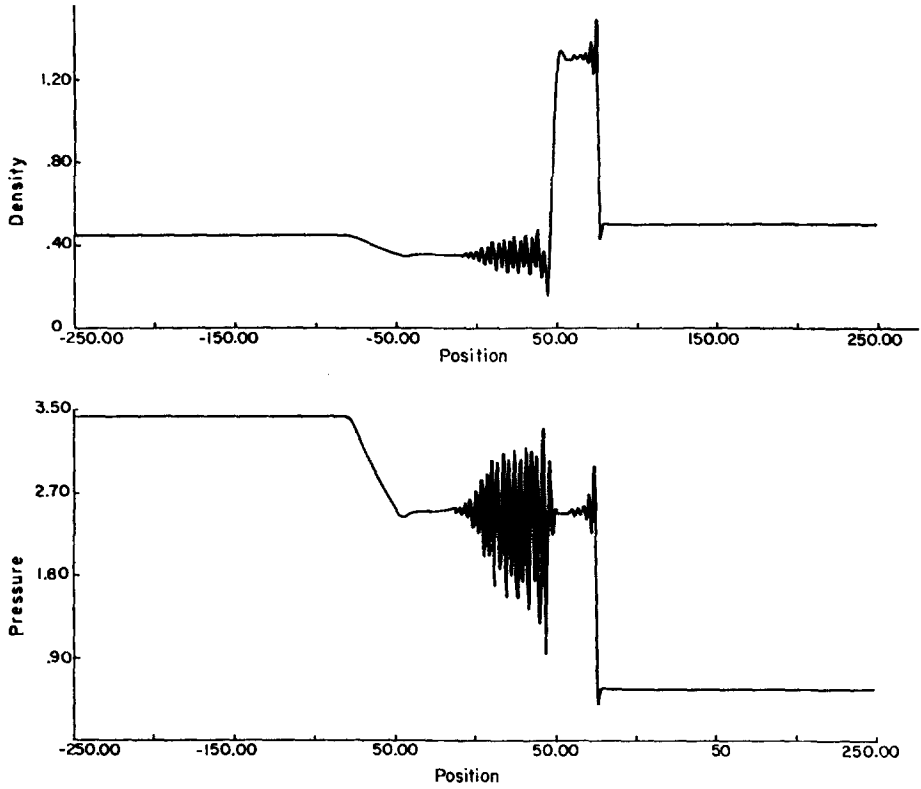

FIG. 5. Pressure and density profile for the Riemann problem after 188 ΔT ($T = 30.138$) using σ and ω not satisfying Eq. (3.5).

which means that the velocity vector depends only on the radius; i.e., $\mathbf{v} = \Omega r$ defines solid body rotation (in our problem centered at $(x_0, y_0) = (30,30)$). If the components of (7.2) are differentiated with respect to x and y respectively we see that one may write (7.1) in conservation form

$$r_t + (ur)_x + (vr)_y = 0 \tag{7.3}$$

since the velocity field is divergence free. The distribution $r(x, y, t)$ is prescribed at $t = 0$ to be a right circular cone in (r, x, y) space centered at (37, 37) with base radius of five and $\Delta x = \Delta y = 1$. Equation (7.3) subject to (7.2) states that the total variation of $r(t)$, $Dr(t)/Dt$, along circles with radius centered at (x_0, y_0), vanishes, i.e., r is just uniformly rotated with period $t = 2\pi/\Omega$. In our computation the mesh size is 60×60 while the cone height is one.

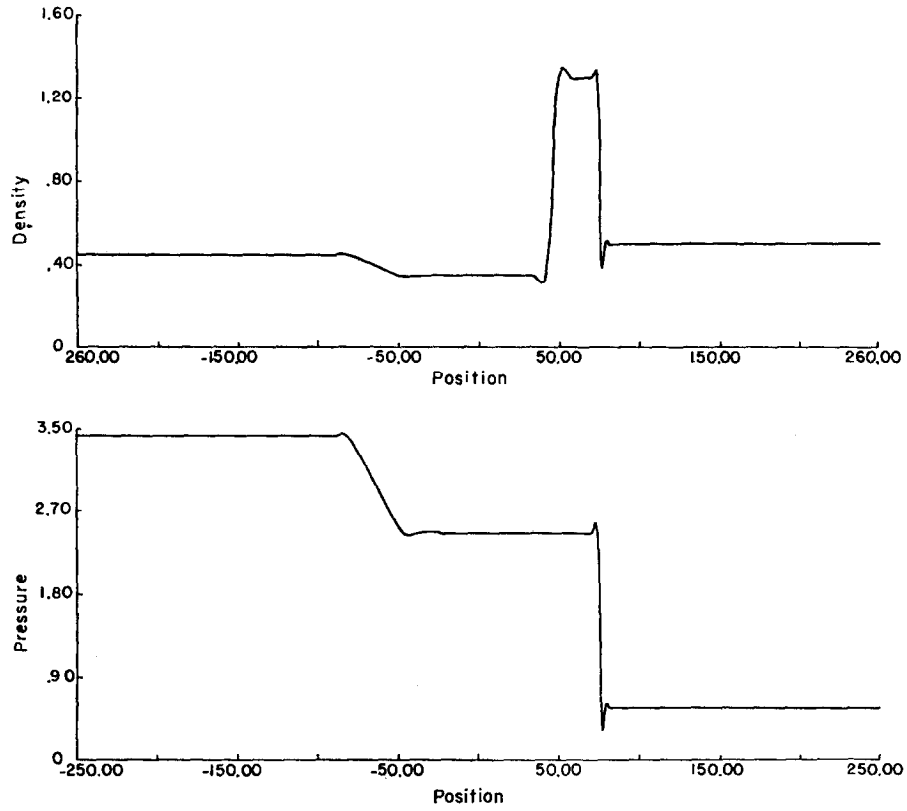

FIG. 6. Pressure and density profile for the Riemann problem after 163 ΔT ($T = 30.065$) using σ and ω satisfying Eq. (3.5). The rarefaction wave propagates to the left; the contact discontinuity is located at $x = 50$; the shock propagates to the right with an error less than one per cent of the theoretical shock speed.

Table II is a summary of the computations performed for this cone problem. In problem 1, the first order scheme is defined by the operator (4.5) while for problems 2, 3 and 4, the second order scheme is defined by the operator (4.8). The third order scheme is given by (4.20) with L defined by system (2.14) and (2.15). The value of ω in problem 5 did not satisfy the stability condition (3.5); it was kept constant. For problems 6, 7, and 8, the local value of ω satisfies $\omega = 4\sigma^2 - \sigma^4 + \epsilon$ with $\epsilon = 0.01$. The value of σ is equal to $\Delta t/\Delta \mid \mathbf{v} \mid$ with $\mid \mathbf{v} \mid = (u^2 + v^2)^{1/2}$. The components of drift of the vertex of the cone in the $x(y)$ direction equals the $x(y)$ position of the vertex computed by the difference method minus the $x(y)$ position of the vertex given by the exact solution.

TABLE II
Summary of Computations for Cone Problem

Problem	Method	Rotations traversed	Computed vertex amplitude	Vertex Drift x-direction	Vertex Drift y-direction	No. of integration cycles
1	first order	1/4	0.07856	6.28699	2.07856	150
2	second order	1	0.98935	1.65263	−2.51629	600
	$\sigma_{max} = 1/6$					
3	second order	1	0.98363	1.55743	−2.29384	300
	$\sigma_{max} = 1/3$	2	0.82304	2.37585	−3.70616	600
4	second order	2	0.79365	2.25867	−3.41507	240
	$\sigma_{max} = 5/6$					
5	third order	1	1.15205	0.33053	−0.29569	600
	$\omega = \text{const} = 0.01$					
	$\sigma_{max} = 1/6$					
6	third order	1	1.03803	0.26070	−0.34469	600
	$\sigma_{max} = 1/6$					
7	third order	1	0.99707	−0.24833	−0.36810	300
	$\sigma_{max} = 1/3$	2	0.89400	−0.24059	−0.45386	600
8	third order	2	0.81353	−0.33365	−0.45469	350
	$\sigma_{max} = 4/7$					
9	third order	≈ 2	unstable			
	$\sigma_{max} = 4/6$					
10	third order	5/12	unstable			
	$\sigma_{max} = 5/6$					
11	third order	1/2	unstable			150
	$\omega = \text{const} = 0.01$	1				300
	$\sigma_{max} = 1/3$					

We see how poorly first order methods compare with second or third order methods in the amplitude and phase of the solution. The most striking difference between second and third order accuracy is in the computation of the phase of the solution. The position of the vertex is within one half mesh width in both the x and y directions for the third order calculation, but is two to three mesh widths from the exact position in both the x and y directions for the second order calculation. For both second and third order schemes, increasing the time step Δt with fixed space step increases σ period. This results in greater dissipation in the third order difference scheme. Increasing Δt also increases the artificial viscosity in the second order method [see [5]] and, therefore, the greater smoothing reduces the maximum amplitude of $r(x, y, t)$.

Problems 9 and 10 went unstable for the values of σ indicated. Hence, one obtains an approximate upper bound for σ, which gives an approximate upper bound for an allowable time step.

The remaining figures are labeled as to problem number, which corresponds to the problems given in the table on the preceding page. The figures show the overall behavior of the various methods and give means for a quick comparison between the methods. The contour lines, at each instant of time, define values $r(x, y) = \text{const}$, the values of which lie between 0.05 and 0.95. For clarity, the snapshot of the solution at the latest time has been shifted by an amount d, which is indicated on the figures.

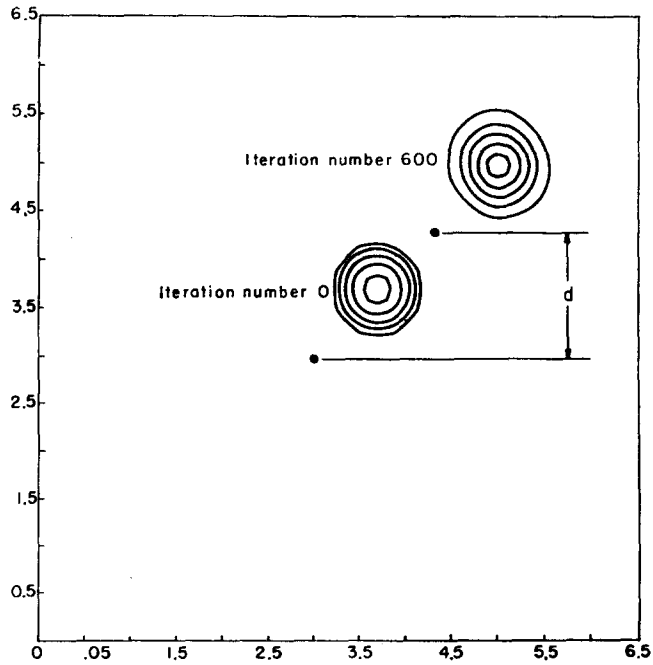

FIG. 7. Problem 2—Second order method with $\sigma_{\max} = 1/6$ and computed vertex amplitude = 0.989; the exact value is 1.

The scheme (4.20) required approximately 4 seconds per sweep while the second order method (4.8) (alternate sweeps were computed using first $L(A/2)\, L(B)\, L(A/2)$ then $L(B/2)\, L(A)\, L(B/2)$, etc., rather than $L(A/2)\, L(B)\, L(A) \cdots L(A)\, L(B)\, L(A/2)$) required approximately 1 1/3 seconds. By comparing the numerical results in the above table, it appears that the mesh ratio, λ, for third order methods can be increased by a factor of three over the second order method. Comparable errors in the amplitude of the solution are obtained with the two methods but a clear superiority in the phase of the solution is achieved with (4.20).

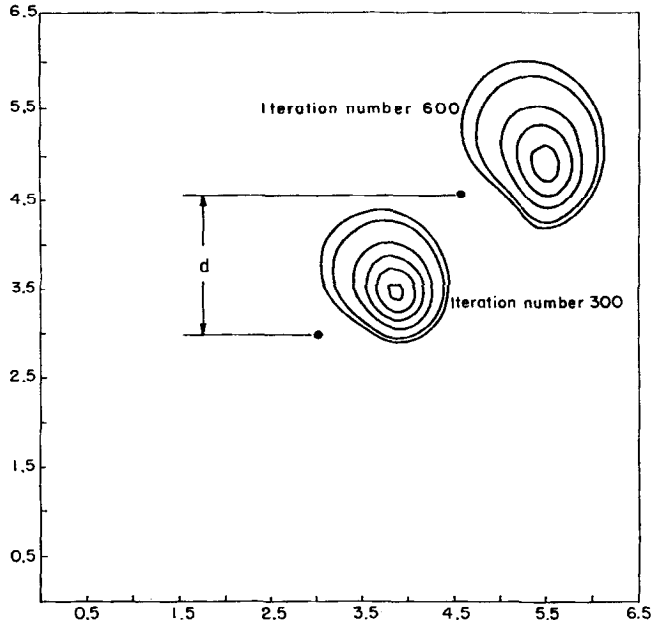

FIG. 8. Problem 3—Same initial data and method as in Fig. 7 but with $\sigma_{max} = 1/3$; after 300 cycles computed vertex amplitude equals 0.983; after 600 cycles amplitude equals 0.823.

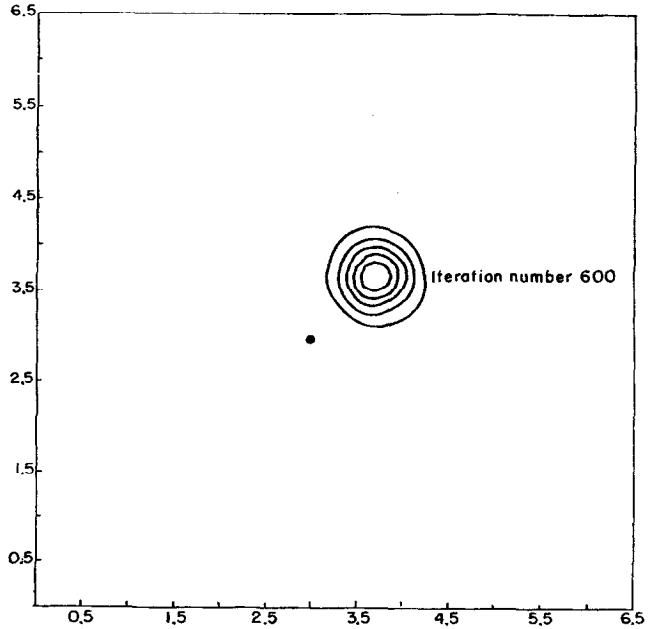

FIG. 9. Problem 6—Same initial data as in Fig. 7; third order method with $\sigma_{max} = 1/6$; ω is variable and is computed from Eq. (3.5). The amplitude is 1.03 after 600 cycles.

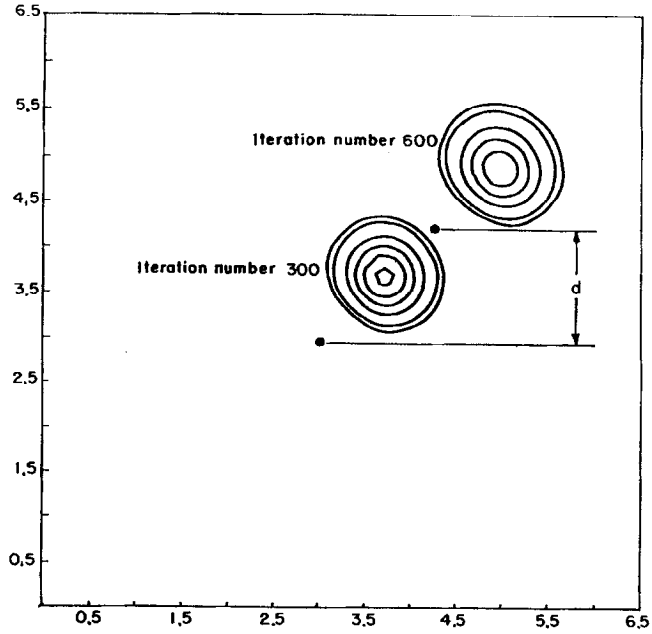

FIG. 10. Problem 7—Same as Fig. 9, but with $\sigma_{max} = 1/3$. The amplitude after 300 cycles is 0.997 and after 600 cycles is 0.894.

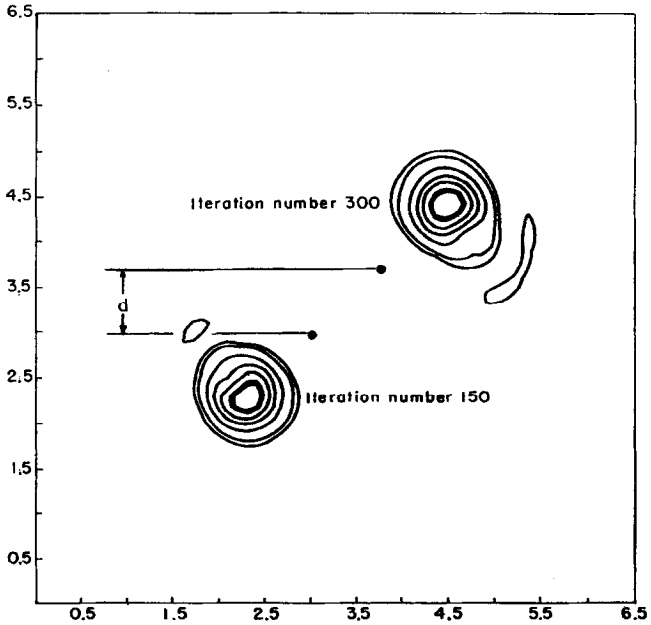

FIG. 11. Problem 11—Same initial data as in Fig. 7; third order method with $\omega = 0.01$ and $\sigma_{max} = 1/3$. Calculation does not satisfy stability condition (3.5). Eddies are forming while the amplitude increases—calculation eventually goes unstable.

Our tentative conclusion, subject to additional numerical tests is that (4.20), using a more coarse mesh, may be as economical as a second order calculation on a fine mesh while still giving superior numerical results. In addition, it is felt that the methods presented would be best suited for problems with smooth solutions rather than shocked flows even though accurate discontinuous solutions can be obtained.

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